ELEMENTS OF ORIENTATION

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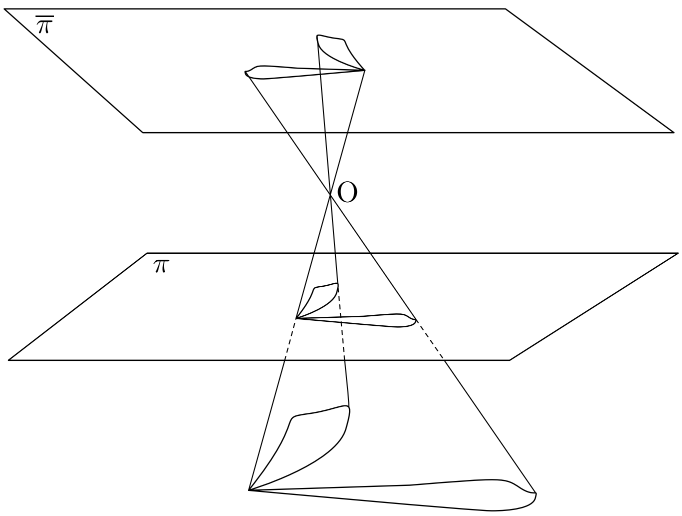
Basics

fig. 1



O: Projection center

$\mathrm{d}(\mathrm{O}, \overline{\pi}) = d(\mathrm{O}, \pi) = \mathrm{f}$,

f: Projection distance,
also focal length

The image is formed on the plane $\overline{\pi}$ but for the formulation of the geometric relations we may suppose that it is formed on the plane π. Some relations as well as the images themselves become clearer when drawn that way.

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fig. 2

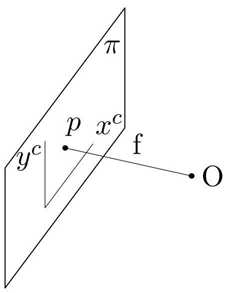
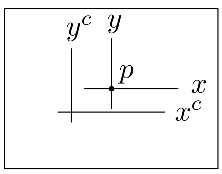


fig. 3



O: Projection center

f: Focal length

p: Principal point

(x^c, y^c): Image system

(x, y): Principal point system

Therefore, (x_p^c, y_p^c) are the coordinates of p, that is, of the origin of the (x, y) system, in the (x^c, y^c) system. The (x, y) system is by definition that which is parallel to the (x^c, y^c) system and centered at p.

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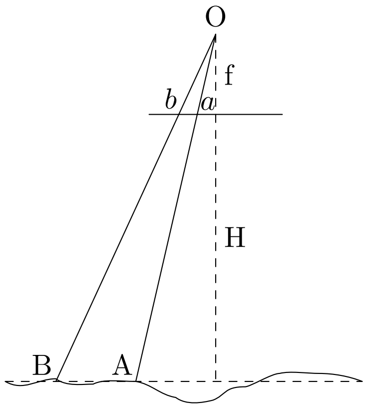
fig. 4



f: Focal length

H: Flight height above ground

a, b: Image points of A and B

$ab = \dfrac{\mathrm{f}}{\mathrm{H}}\,\mathrm{AB}$. Therefore, $\dfrac{\mathrm{f}}{\mathrm{H}}$ is the scale of the image.

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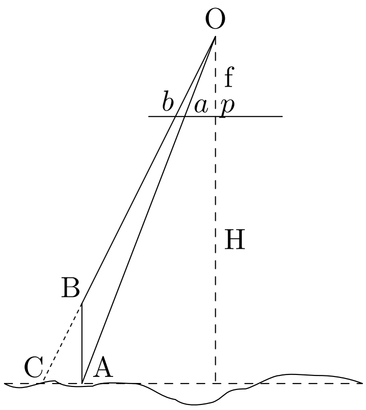
fig. 5



p: Principal point

a, b: Image points of A and B

$\mathrm{AB} = \Delta\mathrm{Z}$

$pa = r_a, \quad pb = r_b, \quad ab = \Delta r$

The image of B is the same as that of C and so $ab = \dfrac{\mathrm{f}}{\mathrm{H}}\,\mathrm{AC}$. But

$$\frac{\mathrm{AC}}{\mathrm{BA}} = \frac{r_b}{\mathrm{f}} \quad \Longrightarrow \quad \mathrm{AC} = \Delta\mathrm{Z}\,\frac{r_b}{\mathrm{f}} \approx \Delta\mathrm{Z}\,\frac{r_a}{\mathrm{f}}$$

The last approximation being made because we are supposing $\Delta\mathrm{Z}$ to be small with respect to H. Hence,

$$\Delta r = \frac{\mathrm{f}}{\mathrm{H}}\Delta\mathrm{Z}\frac{r}{\mathrm{f}} = \frac{r}{\mathrm{H}}\,\Delta\mathrm{Z}$$

which means

$$\Delta x = \frac{x}{\mathrm{H}}\,\Delta\mathrm{Z} \qquad \Delta y = \frac{y}{\mathrm{H}}\,\Delta\mathrm{Z}$$

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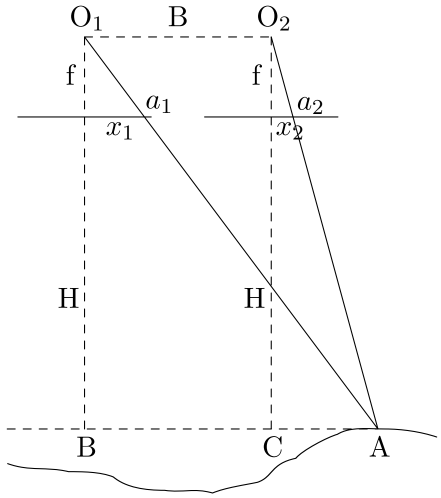
fig. 6



$\overrightarrow{\mathrm{O_1O_2}}$: Direction of x axis in the photographs

$\mathrm{B} = \mathrm{O_1O_2}$: Base

a_1, a_2: Image points of A

(x_1, y_1), (x_2, y_2): Photo coord. of a_1 and a_2

$p = x_1 - x_2$: Parallax

With the x, y axes so chosen, $y_1 = y_2$ and

$$p = x_1 - x_2 = \frac{\mathrm{f}}{\mathrm{H}}\mathrm{BA} - \frac{\mathrm{f}}{\mathrm{H}}\mathrm{CA} = \frac{\mathrm{f}}{\mathrm{H}}(\mathrm{BA} - \mathrm{CA}) = \frac{\mathrm{f}}{\mathrm{H}}\mathrm{B} = \mathrm{f}\frac{\mathrm{B}}{\mathrm{H}}$$

For a small movement in Z of the point A,

$$\Delta p \approx \frac{\mathrm{d}p}{\mathrm{dZ}}\,\Delta\mathrm{Z} = -\frac{\mathrm{fB}}{\mathrm{H}^2}\,\Delta\mathrm{Z} = -\frac{\mathrm{B}}{\mathrm{H}}\frac{\mathrm{f}}{\mathrm{H}}\,\Delta\mathrm{Z}$$

Collecting all the formulas relative to small displacements,

Because of ΔX, ΔY: $\quad \Delta x_1 = \Delta x_2 = \dfrac{\mathrm{f}}{\mathrm{H}}\,\Delta\mathrm{X} \qquad \Delta y_1 = \Delta y_2 = \dfrac{\mathrm{f}}{\mathrm{H}}\,\Delta\mathrm{Y}$

Because of ΔZ: $\quad \Delta x_1 = \dfrac{x_1}{\mathrm{H}}\Delta\mathrm{Z} \qquad \Delta x_2 = \dfrac{x_2}{\mathrm{H}}\Delta\mathrm{Z}$

$$\Delta y_1 = \Delta y_2 = \frac{y}{\mathrm{H}}\Delta\mathrm{Z}$$

$$\Delta p = \Delta(x_1 - x_2) = \Delta x_1 - \Delta x_2 = -\frac{\mathrm{B}}{\mathrm{H}}\frac{\mathrm{f}}{\mathrm{H}}\,\Delta\mathrm{Z}$$

If the photographs have a covering of 60% it means that the same elements appear in one photograph with a displacement of approximately $0.4l$ with respect to the other, where l is the length of the x-side of the photograph. If the focal length equals the half-diagonal of the photo, as in the standard analogical flights,

$$p = x_1 - x_2 \approx 0.4l \approx 0.4\sqrt{2}\mathrm{f} \approx 0.56\mathrm{f}$$

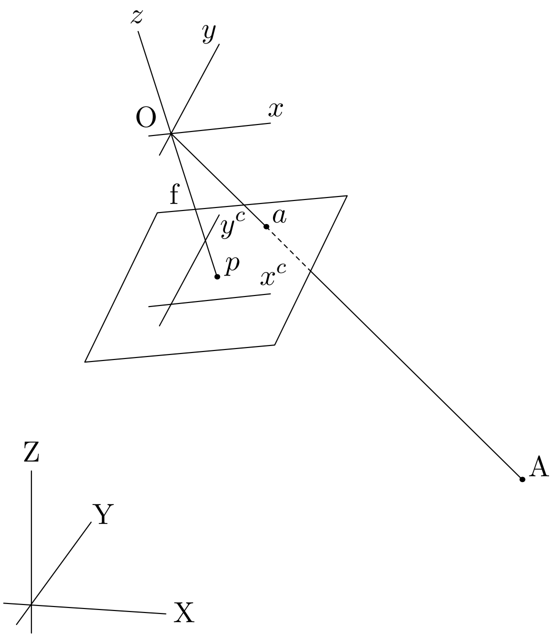
and $p/\mathrm{f} = \mathrm{B}/\mathrm{H}$ (this is most easily seen if the point A is just below $\mathrm{O_2}$), so

$$\frac{\mathrm{B}}{\mathrm{H}} \approx 0.56$$

This means that a displacement in Z has a visible effect on the photo coordinates just 0.56 of that of a displacement in X or in Y.

Theory of one photograph (without distortion)

fig. 7



XYZ: Object coordinate system
p: Image principal point
f: Focal length
xyz: Principal system
$x^c y^c$: Image system
A: object point
a: Image points of A

Let $(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O})$ be the coordinates of the projection center in the object system. The coordinates of A in a system parallel to that of the object and centered at O are

$$(\mathrm{X_A} - \mathrm{X_O}, \mathrm{Y_A} - \mathrm{Y_O}, \mathrm{Z_A} - \mathrm{Z_O})$$

This system can be rotated and thus carried to the x, y, z system. The coordinates of A in this system are therefore

$$\begin{pmatrix} x_\mathrm{A} \\ y_\mathrm{A} \\ z_\mathrm{A} \end{pmatrix} = \mathrm{M} \begin{pmatrix} \mathrm{X_A} - \mathrm{X_O} \\ \mathrm{Y_A} - \mathrm{Y_O} \\ \mathrm{Z_A} - \mathrm{Z_O} \end{pmatrix}$$

i.e.,

$$\begin{aligned} x_\mathrm{A} &= m_{11}(\mathrm{X_A} - \mathrm{X_O}) + m_{12}(\mathrm{Y_A} - \mathrm{Y_O}) + m_{13}(\mathrm{Z_A} - \mathrm{Z_O}) \\ y_\mathrm{A} &= m_{21}(\mathrm{X_A} - \mathrm{X_O}) + m_{22}(\mathrm{Y_A} - \mathrm{Y_O}) + m_{23}(\mathrm{Z_A} - \mathrm{Z_O}) \\ z_\mathrm{A} &= m_{31}(\mathrm{X_A} - \mathrm{X_O}) + m_{32}(\mathrm{Y_A} - \mathrm{Y_O}) + m_{33}(\mathrm{Z_A} - \mathrm{Z_O}) \end{aligned}$$

We will use bold letters to represent vectors: $\mathbf{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})$. When operated with matrices they will always be column vectors. With this notation the previous matrix equation can be written as

$$\mathbf{x}_\mathrm{A} = \mathrm{M}(\mathbf{X}_\mathrm{A} - \mathbf{X}_\mathrm{O})$$

The points A and a are in line with the projection center O. Therefore, their coordinates in any system centered at this point will be proportional. This is known as **colinearity condition**. In particular, in the x, y, z system

$$\frac{x_a}{x_\mathrm{A}} = \frac{y_a}{y_\mathrm{A}} = \frac{z_a}{z_\mathrm{A}}$$

We know $z_a = -\mathrm{f}$, whence

$$x_a = -\mathrm{f}\,\frac{x_\mathrm{A}}{z_\mathrm{A}} \qquad y_a = -\mathrm{f}\,\frac{y_\mathrm{A}}{z_\mathrm{A}} \qquad z_a = -\mathrm{f} \tag{1}$$

Substituting x_A, y_A and z_A by their expression in terms of $\mathrm{X_A}$, $\mathrm{X_O}$, etc., we find

$$\begin{aligned}
x_a &= -\mathrm{f}\,\frac{m_{11}(\mathrm{X_A}-\mathrm{X_O}) + m_{12}(\mathrm{Y_A}-\mathrm{Y_O}) + m_{13}(\mathrm{Z_A}-\mathrm{Z_O})}{m_{31}(\mathrm{X_A}-\mathrm{X_O}) + m_{32}(\mathrm{Y_A}-\mathrm{Y_O}) + m_{33}(\mathrm{Z_A}-\mathrm{Z_O})} \\
y_a &= -\mathrm{f}\,\frac{m_{11}(\mathrm{X_A}-\mathrm{X_O}) + m_{12}(\mathrm{Y_A}-\mathrm{Y_O}) + m_{13}(\mathrm{Z_A}-\mathrm{Z_O})}{m_{31}(\mathrm{X_A}-\mathrm{X_O}) + m_{32}(\mathrm{Y_A}-\mathrm{Y_O}) + m_{33}(\mathrm{Z_A}-\mathrm{Z_O})} \\
z_a &= -\mathrm{f}
\end{aligned} \tag{2}$$

In these equations we see that the parameters defining the coordinates in the principal system of the image point a of an object point A are:

$(\mathrm{X_A}, \mathrm{Y_A}, \mathrm{Z_A})$: The coordinates of A in the object system

$(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O})$: The coordinates of the projection center in the object system

$(\Omega, \Phi, \mathrm{K})$: Three parameters defining a rotation matrix

f: The projection distance

Here the letters Ω, Φ, K simply symbolize three parameters defining a rotation matrix; they needn't be three consecutive rotations around the X, Y and Z axes respectively. They even needn't be rotations at all.

In order to get coordinates in the image system two more parameters are needed:

(x_p^c, y_p^c): Coordinates of the principal point in the image system

These parameters define the position of the center of the image system with respect to the principal point (not the other way round). The coordinates of a in the image system are

$$x_a^c = x_a + x_p^c \qquad y_a^c = y_a + y_p^c$$

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fig. 8

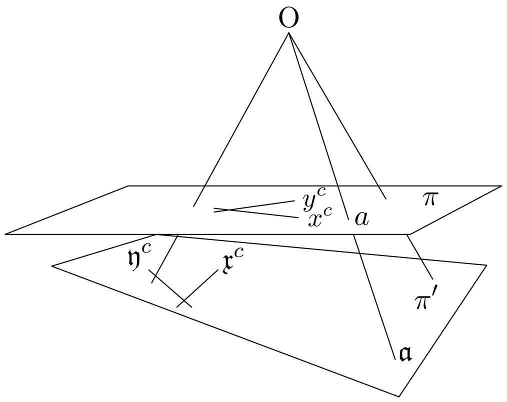


fig. 9

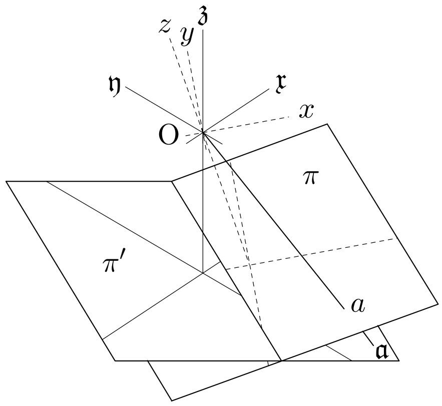


Figure 8 displays two photographs with a common projection center. If the parameters $\Omega, \Phi, \mathrm{K}, \mathrm{f}, x^c, y^c$ of both of them are known then, knowing the coordinates of an image point a in one of the photos, the coordinates of the image point in the other photo, $\mathfrak{a}$, may be computed. The relevant fact is that the coordinates $(\mathrm{X_A}, \mathrm{Y_A}, \mathrm{Z_A})$ of the object point need not be known. The reason is that, provided the projection center O is kept fixed, the projective rays do not change.

This fact allows the analytical computation of a faked image, the projection center of which is the same as that of the image we have and where the other parameters are chosen freely by us. The coordinates used for this transformation are the ones in the principal system. With respect to these coordinates (where there is no x^c, y^c), the two images differ in the rotation matrix M and in the focal length f:

Image	*Parameters*	*Coordinate system*	*Image point*
π	M, f	x, y, z	a
π'	$\mathfrak{M}$, $\mathfrak{f}$	$\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$	$\mathfrak{a}$

The coordinates of $\mathfrak{a}$ in the system $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ are computed by means of the colinearity equations, projecting the point a as if it were the object point. That is, we take image π as the object and the x, y, z coordinate system as the object system, and project a into the image π'.

Proceeding exactly as in the derivation of the colinearity equations, we rotate the object system in order to carry it to the image system; that is, we rotate x, y, z to the position $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, and find the coordinates of the object point, a, in the later system:

$$\begin{pmatrix} \mathfrak{x}_a \\ \mathfrak{y}_a \\ \mathfrak{z}_a \end{pmatrix} = \mathrm{N} \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$$

We recall that $z_a = -\mathrm{f}$. So

$$\begin{aligned} \mathfrak{x}_a &= n_{11} x_a + n_{12} y_a - n_{13} \mathrm{f} \\ \mathfrak{y}_a &= n_{21} x_a + n_{22} y_a - n_{23} \mathrm{f} \\ \mathfrak{z}_a &= n_{31} x_a + n_{32} y_a - n_{33} \mathrm{f} \end{aligned}$$

We apply the colinearity condition:

$$\frac{\mathfrak{x}_\mathfrak{a}}{\mathfrak{x}_a} = \frac{\mathfrak{y}_\mathfrak{a}}{\mathfrak{y}_a} = \frac{\mathfrak{z}_\mathfrak{a}}{\mathfrak{z}_a}, \qquad \mathfrak{z}_\mathfrak{a} = -\mathfrak{f}$$

and finally

$$\begin{aligned}
\mathfrak{x}_\mathfrak{a} &= -\mathfrak{f}\,\frac{n_{11}x_a + n_{12}y_a - n_{13}\mathrm{f}}{n_{31}x_a + n_{32}y_a - n_{33}\mathrm{f}} \\
\mathfrak{y}_\mathfrak{a} &= -\mathfrak{f}\,\frac{n_{21}x_a + n_{22}y_a - n_{23}\mathrm{f}}{n_{31}x_a + n_{32}y_a - n_{33}\mathrm{f}} \\
\mathfrak{z}_\mathfrak{a} &= -\mathfrak{f}
\end{aligned}$$

Removing the subindices:

$$\begin{aligned}
\mathfrak{x} &= -\mathfrak{f}\,\frac{n_{11}x + n_{12}y - n_{13}\mathrm{f}}{n_{31}x + n_{32}y - n_{33}\mathrm{f}} \\
\mathfrak{y} &= -\mathfrak{f}\,\frac{n_{21}x + n_{22}y - n_{23}\mathrm{f}}{n_{31}x + n_{32}y - n_{33}\mathrm{f}} \\
\mathfrak{z} &= -\mathfrak{f}
\end{aligned}$$

There remains to find the rotation matrix N. It is the one that carries x, y, z to $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$. This result may be achieved by first removing the rotation M, thus orientating x, y, z as the original object system, and therefrom applying $\mathfrak{M}$:

$$\boldsymbol{\mathfrak{x}} = \mathfrak{M}\mathrm{M}^{-1}\mathbf{x} \qquad \Longrightarrow \qquad \mathrm{N} = \mathfrak{M}\mathrm{M}^{-1}$$

Example: Let an image have $\mathrm{f} = 152.015\,\mathrm{mm}$ and

$$\mathrm{M} = \begin{pmatrix} 0.9964 & 0.0828 & -0.0187 \\ -0.0829 & 0.9966 & 0.0005 \\ 0.0187 & 0.0011 & 0.9998 \end{pmatrix}$$

It is wanted to create the image corresponding to the same projection distance and

$$\mathfrak{M} = \begin{pmatrix} 0.9923 & -0.1207 & -0.0287 \\ 0.1204 & 0.9927 & -0.0104 \\ 0.0297 & 0.0069 & 0.9995 \end{pmatrix}$$

Find the coordinates in the new image of the point $(102.864, 66.091)$.

The rotation matrix from the first image to the second is

$$\mathrm{N} = \mathfrak{M}\mathrm{M}^{-1} = \begin{pmatrix} 0.9793 & -0.2026 & -0.0103 \\ 0.2024 & 0.9793 & -0.0071 \\ 0.0115 & 0.0049 & 0.9999 \end{pmatrix}$$

The coordinates of the original point, a, in the principal system of the second image are

$$\begin{pmatrix}\mathfrak{x}_a\\ \mathfrak{y}_a\\ \mathfrak{z}_a\end{pmatrix} = \mathrm{N}\begin{pmatrix}102.864\\ 66.091\\ -152.015\end{pmatrix} = \begin{pmatrix}88.905\\ 86.613\\ -150.489\end{pmatrix}$$

and the searched values

$$\mathfrak{x} = -152.015\,\frac{88.905}{-150.489} = 89.807$$

$$\mathfrak{y} = -152.015\,\frac{86.613}{-150.489} = 87.491$$

A particular case of these transformations is the one $\mathfrak{M} = \mathrm{I}$. Another particular case consists in modifying just the κ angle. This last one does not change the projection plane π, so it is just a plane transformation:

$$\mathfrak{x} = \cos\kappa\, x - \sin\kappa\, y$$
$$\mathfrak{y} = \sin\kappa\, x + \cos\kappa\, y$$

The way the second image is actually computed is by the resampling technique. This is explained in the paper "Orthophotographs".

We come now to the linearization of the colinearity equations. When an adjustment is carried out the solution is found as usual by iterating, starting form an approximate solution. At each iteration the exact equations are replaced by linear ones.

Unless we are calibrating the values x_p^c, y_p^c, the working coordinates will always be the ones referred to the principal system, i.e., $x_a = x_a^c - x_p^c$ and $y_a = y_a^c - y_p^c$, and this we shall suppose.

Let the approximate values be f_0, $(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O})_0$, M_0, $(\mathrm{X_A}, \mathrm{Y_A}, \mathrm{Z_A})_0$. To these values there correspond two values (x_{a0}, y_{a0}). The values of (x_a, y_a) corresponding to small increments of the approximate values are, within a first order approximation,

$$x_a \approx x_{a0} + \left(\frac{\partial x}{\partial \mathrm{f}}\right)_0 \Delta\mathrm{f} + \cdots + \left(\frac{\partial x}{\partial \mathrm{Z_A}}\right)_0 \Delta\mathrm{Z_A}$$
$$y_a \approx y_{a0} + \left(\frac{\partial y}{\partial \mathrm{f}}\right)_0 \Delta\mathrm{f} + \cdots + \left(\frac{\partial y}{\partial \mathrm{Z_A}}\right)_0 \Delta\mathrm{Z_A}$$

The subindex 0 means that the derivatives are evaluated at the approximate values.

Rotations matrices are composed by multiplying them. So a small rotation added to M_0 results in $\mathrm{M}_\Delta\mathrm{M}_0$. A small rotation matrix is in first order

$$\mathrm{M}_\Delta = \begin{pmatrix}1 & -\kappa & \phi\\ \kappa & 1 & -\omega\\ -\phi & \omega & 1\end{pmatrix}$$

and it is required to find the derivatives of (x_a, y_a) with respect to ω, ϕ and κ.

We will find the derivatives applying the chain rule to equations (1), finding first the derivatives of $(x_\mathrm{A}, y_\mathrm{A}, z_\mathrm{A})$:

$$\mathbf{x}_\mathrm{A} = \mathrm{M}_\Delta \mathrm{M}_0 \mathbf{X}_{\mathrm{A}0} = \mathrm{M}_\Delta \mathbf{x}_{\mathrm{A}0}$$

$$\begin{aligned} x_\mathrm{A} &= x_{\mathrm{A}0} - \kappa y_{\mathrm{A}0} + \phi z_{\mathrm{A}0} \\ y_\mathrm{A} &= \kappa x_{\mathrm{A}0} + y_{\mathrm{A}0} - \omega z_{\mathrm{A}0} \\ z_\mathrm{A} &= -\phi x_{\mathrm{A}0} + \omega y_{\mathrm{A}0} + z_{\mathrm{A}0} \end{aligned}$$

Hence

$$\begin{array}{lll} \partial x_\mathrm{A}/\partial\omega = 0 & \partial x_\mathrm{A}/\partial\phi = z_\mathrm{A} & \partial x_\mathrm{A}/\partial\kappa = -y_\mathrm{A} \\ \partial y_\mathrm{A}/\partial\omega = -z_\mathrm{A} & \partial y_\mathrm{A}/\partial\phi = 0 & \partial y_\mathrm{A}/\partial\kappa = x_\mathrm{A} \\ \partial z_\mathrm{A}/\partial\omega = y_\mathrm{A} & \partial z_\mathrm{A}/\partial\phi = -x_\mathrm{A} & \partial z_\mathrm{A}/\partial\kappa = 0 \end{array}$$

There is no need to write $\Delta\omega$, $\Delta\phi$, $\Delta\kappa$ in place of ω, ϕ, κ, since the later are already the small values representing the increment to M_0. Also, the subindex 0 has been removed, since it is used to indicate the evaluation point, not the derivative itself as a function.

By application of the chain rule we find:

$$\begin{aligned} \frac{\partial x_a}{\partial\omega} &= \frac{\partial x_a}{\partial x_\mathrm{A}}\frac{\partial x_\mathrm{A}}{\partial\omega} + \frac{\partial x_a}{\partial z_\mathrm{A}}\frac{\partial z_\mathrm{A}}{\partial\omega} = 0 + \mathrm{f}\,\frac{x_\mathrm{A}}{z_\mathrm{A}^2} y_\mathrm{A} = -x_a \frac{y_\mathrm{A}}{z_\mathrm{A}} \\ \frac{\partial x_a}{\partial\phi} &= \frac{\partial x_a}{\partial x_\mathrm{A}}\frac{\partial x_\mathrm{A}}{\partial\phi} + \frac{\partial x_a}{\partial z_\mathrm{A}}\frac{\partial z_\mathrm{A}}{\partial\phi} = -\mathrm{f}\,\frac{1}{z_\mathrm{A}} z_\mathrm{A} + \mathrm{f}\,\frac{x_\mathrm{A}}{z_\mathrm{A}^2}(-x_\mathrm{A}) = -\mathrm{f} - \mathrm{f}\,\frac{x_\mathrm{A}^2}{z_\mathrm{A}} = -\mathrm{f} + x_a \frac{x_\mathrm{A}}{z_\mathrm{A}} \\ \frac{\partial x_a}{\partial\kappa} &= \frac{\partial x_a}{\partial x_\mathrm{A}}\frac{\partial x_\mathrm{A}}{\partial\kappa} + \frac{\partial x_a}{\partial z_\mathrm{A}}\frac{\partial z_\mathrm{A}}{\partial\kappa} = -\mathrm{f}\,\frac{1}{z_\mathrm{A}}(-y_\mathrm{A}) + 0 = \mathrm{f}\,\frac{y_\mathrm{A}}{z_\mathrm{A}} = -y_a \end{aligned}$$

and like formulas for y_a. The six derivatives are

$$\begin{array}{lll} \partial x_a/\partial\omega = -x_a \dfrac{y_\mathrm{A}}{z_\mathrm{A}} & \partial x_a/\partial\phi = -\mathrm{f} + x_a \dfrac{x_\mathrm{A}}{z_\mathrm{A}} & \partial x_a/\partial\kappa = -y_a \\ \partial y_a/\partial\omega = \mathrm{f} - y_a \dfrac{y_\mathrm{A}}{z_\mathrm{A}} & \partial y_a/\partial\phi = y_a \dfrac{x_\mathrm{A}}{z_\mathrm{A}} & \partial y_a/\partial\kappa = x_a \end{array}$$

The derivatives with respect to the other parameters are easily found:

$$\frac{\partial x_a}{\partial \mathrm{f}} = -\frac{x_\mathrm{A}}{z_\mathrm{A}} = \frac{x_a}{\mathrm{f}} \qquad \frac{\partial y_a}{\partial \mathrm{f}} = -\frac{y_\mathrm{A}}{z_\mathrm{A}} = \frac{y_a}{\mathrm{f}}$$

$$\frac{\partial x_a}{\partial x_\mathrm{A}} = -\mathrm{f}\,\frac{m_{11} z_\mathrm{A} - m_{31} x_\mathrm{A}}{z_\mathrm{A}^2} = -\frac{m_{11}\mathrm{f} + m_{31} x_a}{z_\mathrm{A}}, \qquad \frac{\partial x_a}{\partial x_\mathrm{O}} = -\frac{\partial x_a}{\partial x_\mathrm{A}}$$

To avoid minus signs we write the derivatives with respect to the coordinates of O, those with respect to the coordinates of A being the opposite ones:

$$\frac{\partial x_a}{\partial x_\mathrm{O}} = \frac{m_{11}\mathrm{f} + m_{31} x_a}{z_\mathrm{A}} \qquad \frac{\partial x_a}{\partial y_\mathrm{O}} = \frac{m_{12}\mathrm{f} + m_{32} x_a}{z_\mathrm{A}}$$

$$\frac{\partial x_a}{\partial z_{\mathrm{O}}} = \frac{m_{13}\mathrm{f} + m_{33}x_a}{z_{\mathrm{A}}}$$

$$\frac{\partial y_a}{\partial x_{\mathrm{O}}} = \frac{m_{21}\mathrm{f} + m_{31}y_a}{z_{\mathrm{A}}} \qquad \frac{\partial y_a}{\partial y_{\mathrm{O}}} = \frac{m_{22}\mathrm{f} + m_{32}y_a}{z_{\mathrm{A}}}$$

$$\frac{\partial y_a}{\partial z_{\mathrm{O}}} = \frac{m_{23}\mathrm{f} + m_{33}y_a}{z_{\mathrm{A}}}$$

When the system of equations where these derivatives appear is solved, the increments $\Delta\mathrm{f}$, etc. are added to the approximate values. For the rotation matrix, the product $\mathrm{M}_\Delta\mathrm{M}_0$ is performed. If the ω, ϕ, κ increments are not very small (for instance, in the first iteration), M_Δ is better computed by an exact formula as a function of ω, ϕ, κ. Any formula which is in first order as the M_Δ shown above is valid. There are several of them; see the paper "Matrices de Rotacion". Numbers 5 and 6 in that paper are the ones requiring the least number of operations.

If the values x_p^c, y_p^c are being calibrated, then $x_a^c = x_a + x_p^c$ is used. All the derivatives remain equal, still using the values x_a, y_a (i.e., not x_a^c, y_a^c), and we further have

$$\frac{\partial x_a^c}{\partial x_p^c} = 1 \qquad \frac{\partial y_a^c}{\partial y_p^c} = 1$$

The units of the object coordinate system and of the image one, either x^c, y^c or the principal system, are independent. All the formulas found so far where numbers in both systems appear, and any others we may find, can be written in the form

$$\frac{a}{b} = \frac{\mathrm{X}}{\mathrm{Y}}$$

where a and b are magnitudes in the image system and X and Y magnitudes in the object system. Note that the former include the focal length: $\mathrm{f} = -z_a$ for all $a \in \pi$.

For this reason, if at the projection plane there is not a film but a pixel matrix, the focal length only has a meaning expressed in pixel units. A second photograph taken with double projection distance and at the same time double pixel size would be identical to a first one. Indeed, the very sentence "double pixel size" is absurd since the units are the pixels; and also because the units are the pixels the second photograph has the same focal length than the first one.

The scale of the image, which we found to be f/H, is expressed as

$$\frac{\mathrm{f}}{\mathrm{H}} = \frac{\textit{image units}}{\textit{object units}}$$

or by its inverse.

Transforming the pixels to millimeters or microns, as is usually done, is a reminiscent of working with those units with analogical cameras. It is doubly absurd if we take into account that in order to decide whether a given image is adequate for a certain work, the millimeters are transformed back to pixels by means of the pixel size, in order to find the relation *meters per pixel,* which is no other than the original scale in pixel units.

The parameters defining the image coordinates of a point, apart form the object coordinates themselves, are nine:

$$\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \Omega, \Phi, \mathrm{K}, \mathrm{f}, x_p^c, y_p^c$$

They can be divided according to different criteria. In the process of exploiting the image, once the parameters have been computed, the most important division is

$$\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O} \quad \Omega, \Phi, \mathrm{K}, \mathrm{f}, x_p^c, y_p^c$$

because, as we saw, the first three have to remain fixed; they are “locked”, while the others may be varied and the new image computed accordingly.

From the geometric viewpoint the division is

$$\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \Omega, \Phi, \mathrm{f} \quad \kappa, x_p^c, y_p^c$$

Here κ does mean a κ rotation, while Ω and Φ are still symbolic names for two parameters. The motivation for this division is that, from a geometric viewpoint, the image is defined by the first six parameters —i.e., the last three parameters do not actually exist. In order to define a conic projection the position of the projection center is needed along with the orientation of the plane π, which is the parameters Ω, Φ, and the projection distance f.

Finally, when computing the parameters the most relevant division is

$$\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \Omega, \Phi, \mathrm{K} \quad \mathrm{f}, x_p^c, y_p^c$$

for the last three do not vary from one image to another.

Apart from these divisions, the coordinates used will always be the ones in the principal system, x, y; which can also be thought as the parameters x_p^c, y_p^c being zero, or even that they do not exist. The only exception to this is when they are being calibrated.

Models

A model is a set of oriented photographs and object points. For example,

photo$_1$	$(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \mathrm{M}, \mathrm{f})_1$	*point*$_1$	$(\mathrm{X}, \mathrm{Y}, \mathrm{Z})_1$
photo$_2$	$(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \mathrm{M}, \mathrm{f})_2$	*point*$_2$	$(\mathrm{X}, \mathrm{Y}, \mathrm{Z})_2$
...		...	
photo$_n$	$(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \mathrm{M}, \mathrm{f})_n$	*point*$_m$	$(\mathrm{X}, \mathrm{Y}, \mathrm{Z})_m$

Its object system is arbitrary and is called model system. It very often coincides with the principal system of one of its photographs.

Since the object and image systems are independent from each other, we may change the units of one of them without having to do anything to the other. This is the case, for instance, when pixels are transformed to microns, or microns to millimeters ... Within the object, we may transform from meters to kilometers, from kilometers to miles ... A change in units is applied by multiplying all the values by a constant. Since the model system is arbitrary, it is also arbitrary on its units. Therefore, the transformation from the model system to the actual object system is a translation and a rotation plus a scale factor; i.e., a similarity transformation.

The multiplication of all the model coordinates by a constant can be interpreted either as a change in the units, as we have done, or as actually making the model bigger or smaller:

fig. 10

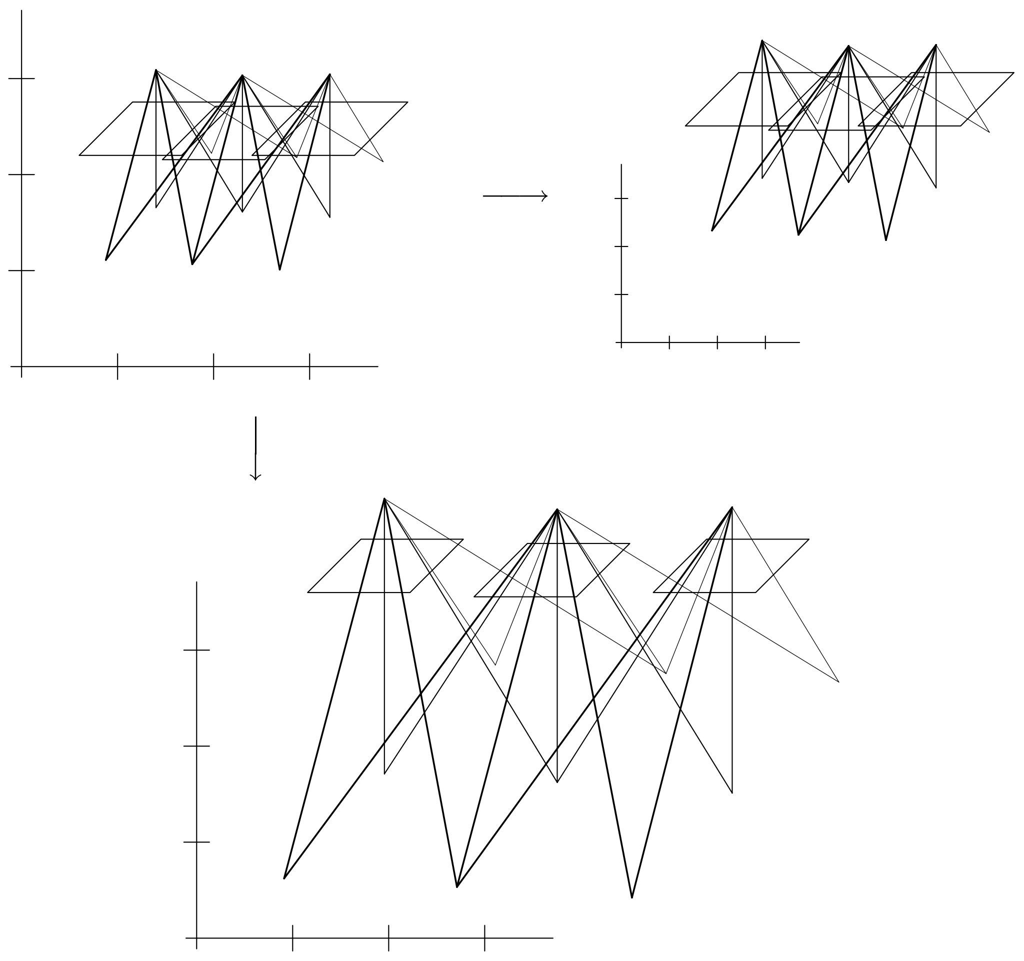
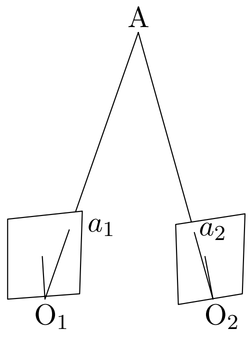


fig. 11



The direct problem within a model consists in knowing the image coordinates of an object point in several photos from the model and therefrom computing its object coordinates.

In the simplest case that the point only appears in two photographs (or it has only been measured in two photographs, or it is only the information from those photographs that is taken into account), we need only find the equations of the two right lines passing through each projection center and its respective image point, and compute the intersection of two lines in space, which is a well known problem.

One of those lines passes through O_1 and a_1 and the other through O_2 and a_2. We first need to find the coordinates in the object system of a_1 and a_2:

$$\mathbf{X}_{a_1} = \mathbf{X}_{\mathrm{O}_1} + \mathrm{M}^{-1}\mathbf{x}_{a_1} \qquad \text{(resp. for } a_2\text{)}$$

Due to imperfections in the image and the measurements, the two right lines as defined by the coordinates of these four points will not intersect in space, but pass very near to each other. It is a typical algebraic problem to find the midpoint of the shortest segment joining the two lines. However, in the by far most usual configuration of the two photographs in photogrammetry, this solution is not better (neither worse) than the midpoint of the shortest *horizontal* segment joining the two lines, that is, the shortest segment with constant Z, and this later solution is more easily computed, as we now show.

For the sake of simplicity and generality let a, b, c and d be the four points, a and b belonging to one line and c and d to the other. For a given z coordinate, the (x, y) coordinates in the first line can be found by interpolation:

$$x_1 = \frac{z - z_a}{z_b - z_a}(x_b - x_a) + x_a, \qquad y_1 = \frac{z - z_a}{z_b - z_a}(y_b - y_a) + y_a$$

and the respective formulas for (x_2, y_2). These formulas can in turn be written as

$$x_1 = m_1 z + n_1, \qquad y_1 = p_1 z + q_1,$$

$$x_2 = m_2 z + n_2, \qquad y_2 = p_2 z + q_2.$$

The square of the horizontal distance between the lines for a certain z is therefore

$$\big((m_1 z + n_1) - (m_2 z + n_2)\big)^2 + \big((p_1 z + q_1) - (p_2 z + q_2)\big)^2.$$

When carrying out the operations this becomes an expression of the form

$$rz^2 + 2sz + t.$$

The value of z for which this is minimum is

$$z = \frac{s}{r}.$$

With this value of z the points (x_1, y_1) and (x_2, y_2) are found and the solution is the midpoint thereof.

Example:

$$\mathrm{O}_1 = (0,0,0) \qquad \mathrm{O}_2 = (411.48, 240.99, 1.02) \qquad \mathrm{f} = 41.883$$

$$\mathrm{M}_1 = \begin{pmatrix} 0.86840 & 0.49583 & 0.00657 \\ -0.49572 & 0.86838 & -0.01351 \\ -0.01240 & 0.00848 & 0.99989 \end{pmatrix} \qquad \mathrm{M}_2 = \begin{pmatrix} 0.86855 & 0.49552 & 0.00868 \\ -0.49537 & 0.86855 & -0.01548 \\ -0.01521 & 0.00915 & 0.99984 \end{pmatrix}$$

$$a_1 = (-0.189, 0.638) \qquad a_2 = (-19.590, 0.502)$$

We first have to calculate the model coordinates of the points a_1 and a_2, or rather the vectors $\overrightarrow{\mathrm{O}_1 a_1}$ and $\overrightarrow{\mathrm{O}_2 a_2}$, for it is the components of these vectors that appear in the formula of the intersection (as $z_b - z_a$, $x_b - x_a$, etc.).

$$\mathbf{X}_{\overrightarrow{\mathrm{O}_1 a_1}} = \mathrm{M}_1^{-1}\mathbf{x}_{1a_1} = \begin{pmatrix} 0.86840 & -0.49572 & -0.01240 \\ 0.49583 & 0.86838 & 0.00848 \\ 0.00657 & -0.01351 & 0.99989 \end{pmatrix} \begin{pmatrix} -0.189 \\ 0.638 \\ -41.883 \end{pmatrix} = \begin{pmatrix} 0.039 \\ 0.105 \\ -41.888 \end{pmatrix}$$

$$\mathbf{X}_{\overrightarrow{\mathrm{O}_2 a_2}} = \mathrm{M}_2^{-1}\mathbf{x}_{2a_2} = \begin{pmatrix} 0.86855 & -0.49537 & -0.01521 \\ 0.49552 & 0.86855 & 0.00915 \\ 0.00868 & -0.01548 & 0.99984 \end{pmatrix} \begin{pmatrix} -19.590 \\ 0.502 \\ -41.883 \end{pmatrix} = \begin{pmatrix} -16.627 \\ -9.654 \\ -42.054 \end{pmatrix}$$

Now we have got the four points, O_1 and a_1, and O_2 and a_2, with coordinates in the model system, and it is required to find the best intersection as it has been explained. Let the four points be renamed a, b, c, d:

$$x_1 = \frac{z - z_a}{z_b - z_a}(x_b - x_a) + x_a = \frac{z}{-41.888}\,0.039 + 0 = -0.0009z$$

$$y_1 = \frac{z - z_a}{z_b - z_a}(y_b - y_a) + y_a = \frac{z}{-41.888}\,0.105 + 0 = -0.0025z$$

$$x_2 = \frac{z - z_c}{z_d - z_c}(x_d - x_c) + x_c = \frac{z - 1.02}{-42.054}(-16.627) + 411.48 = 0.3954z + 411.08$$

$$y_2 = \frac{z - z_c}{z_d - z_c}(y_d - y_c) + y_c = \frac{z - 1.02}{-42.054}(-9.654) + 240.99 = 0.2296z + 240.76$$

Whence

$$x_2 - x_1 = 0.3963z + 411.08 \qquad y_2 - y_1 = 0.2321z + 240.76$$

and

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (0.1571 + 0.0539)z^2 + 2(162.91 + 55.87)z + 411^2 + 241^2$$
$$= 0.2110z^2 + 2 \cdot 218.78z + \text{const.}$$

The minimum for this quantity is achieved at

$$z = \frac{-218.78}{0.2110} = -1037.33$$

From this value of z, substituting in the equations for x_1, y_1, x_2 and z_2, we find

$$(x_1, y_1) = (0.97, 2.59) \qquad (x_2, y_2) = (0.95, 2.62)$$

The mean value is the searched solution:

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z})_\mathrm{A} = (0.96, 2.61, -1037.33)$$

Two models can be joined into a single one; one photo may be added to an existing model, and finally a set of independent photos can be joined to form a new model.

Of these possibilities the more basic one is that of joining two photographs to form a model. Since the coordinate system of a model is arbitrary, it is usually made coincident in position and orientation with the principal system of one of the photographs, *the first* photograph, and the other is *the second* photograph. Therefore

$$(\mathrm{X_O}, \mathrm{Y_O}, \mathrm{Z_O}, \mathrm{M})_1 = (0, 0, 0, \mathrm{I}),$$

where I is the identity matrix.

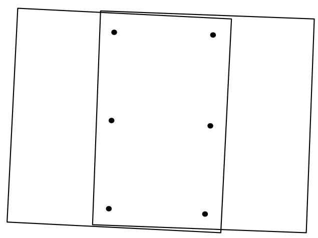
There remains to select the model scale. This is best defined by fixing the distance between the two projection centers, but for practical computational reasons it is the coordinate X_2 which is chosen.

The relative position of the second photograph with respect to the first one is given by its parameters in the model system so chosen —i.e., $(\mathrm{X}_2, \mathrm{Y}_2, \mathrm{Z}_2, \mathrm{M})$. Among these, X_2 has been freely chosen, so there only remain five to calculate:

$$\mathrm{Y}_2, \mathrm{Z}_2, \, \Omega, \Phi, \mathrm{K}$$

These are found by relating the position of image points in one and the other photograph. A minimum of five points measured in both photographs is needed, but in order to get a reliable orientation at least six are needed. These six points are taken at the so called *Von Gruber* locations:

fig. 12



This is a particular case of a very important principle in photogrammetry: **Do never extrapolate** (out of the area covered by the points).

The position of image points can be related to the orientation of one photograph to the other (relative orientation) by the fact that the vectors $\overrightarrow{\mathrm{O_1O_2}}$, $\overrightarrow{\mathrm{O}_1a_1}$ and $\overrightarrow{\mathrm{O}_2a_2}$, as contained in the triangle $\mathrm{AO_1O_2}$, are coplanar. Coplanarity is expressed algebraically by the property of being linearly dependent, which in turns means that the determinant formed by the vectors vanishes:

$$\begin{vmatrix} \overrightarrow{\mathrm{O_1O_2}} \\ \overrightarrow{\mathrm{O}_1a_1} \\ \overrightarrow{\mathrm{O}_2a_2} \end{vmatrix} = 0$$

The three vectors need be expressed in the same coordinate system, of course, and the model system is chosen for that. We have already seen that $\overrightarrow{\mathrm{O}_1a_1}$ in the model system is $\mathrm{M}_1^{-1}\mathbf{x}_{1a_1}$ and resp. for $\overrightarrow{\mathrm{O}_2a_2}$.

Taking the same example as for the intersection of right lines, the condition is

$$\begin{vmatrix} 411.48 & 240.99 & 1.02 \\ 0.039 & 0.105 & -41.888 \\ -16.627 & -9.654 & -42.054 \end{vmatrix} = 0$$

The actual value of this determinant is 26, due to imperfections in the measurements and others.

When computing the orientation, the values of $\mathrm{X_2, Y_2, Z_2}, \Omega, \Phi, \mathrm{K}$, on which the vectors $\overrightarrow{\mathrm{O_1O_2}}$ and $\overrightarrow{\mathrm{O}_2a_2}$ depend, are the unknowns. With five measured points we have a system of five equations (not linear) with five unknowns, that may be solved; but as I said, we had better have at least six points.

Another usual selection criterion for the fixed parameters and the unknowns is to take both $\mathrm{O_1}$ and $\mathrm{O_2}$ fixed, as well as Ω_1, and let the unknowns be Φ_1, K_1 and $\Omega_2, \Phi_2, \mathrm{K}_2$.

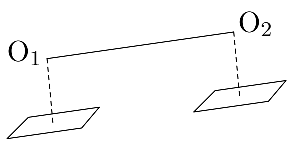
Either when more than two photographs are joined, or when any of the other possibilities is performed, model-photo or model-model, we will have to adjust more than two photos at once, and the coplanarity condition is no longer applicable, at least not as easily as in the two-photo case.

However, the colinearity equations apply equally to two photographs than to more of them. The equations are established, one for x and one for y for each image point, the unknowns being the six orientation parameters of that photograph, $(\mathrm{X, Y, Z}, \Omega, \Phi, \mathrm{K})$, and the $(\mathrm{X, Y, Z})$ coordinates of the object point. There are more equations than in the coplanarity method, but there are as well more unknowns.

Seven parameters have to be fixed as when applying coplanarity. This follows from the fact that it is a geometric need, not something depending upon the method of calculation.

As it was shown some pages back, given an image and its orientation parameters, a different image can be analytically (or mechanically) generated with other parameters chosen at will, with the only restriction that the projection center remain the same. This has a very profitable application within two-photo models: two new images can be generated with the orientation parameters so chosen that they satisfy the requirements of the normal case, namely, $\mathrm{M}_1 = \mathrm{M}_2$ and the x axes of both photographs coincident ad parallel to the base $\mathrm{O}_1\mathrm{O}_2$.

fig. 13



There is an infinity of solutions, differing in a rotation around the axis that joins the two projection centers. The solution which modifies the least the orientation of the first photograph is as follows: Let $\mathfrak{m}_1$ be a unitary vector in the direction and sense of $\overrightarrow{\mathrm{O}_1\mathrm{O}_2}$, and let $\mathbf{s}$ be the third row of M_1, considered as vector. Then

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \\ \mathfrak{m}_3 \end{pmatrix}, \qquad \mathfrak{m}_2 = \frac{\mathbf{s} \wedge \mathfrak{m}_1}{|\mathbf{s} \wedge \mathfrak{m}_1|}, \qquad \mathfrak{m}_3 = \mathfrak{m}_1 \wedge \mathfrak{m}_2$$

is the rotation matrix of the two new images. Hence, those that pass from M_1 and M_2 to $\mathfrak{M}$ are $\mathfrak{M}\mathrm{M}_1^{-1}$ and $\mathfrak{M}\mathrm{M}_2^{-1}$ respectively. The new images are called epipolar.

All the possible solutions have in common the row $\mathfrak{m}_1$. Then we can peek any vector in place of $\mathbf{s}$, and compute $\mathfrak{m}_2$ and $\mathfrak{m}_3$ with the same formulas. In case that the photographs ought to be parallel to the XY plane, as in aerial photogrammetry, the best solution is that which generates the most vertical photos, which is also the easiest one to compute: in place of $\mathbf{s}$ the vector $(0, 0, 1)$ is taken.

Outer Orientation

The outer orientation of a photograph is composed of the coordinates of the projection center O and the rotation matrix M. Therefore, a set of photographs with their outer orientation together with a set of object points is a model where the coordinate system is the object system.

The outer orientation of a block is solved by establishing the system of colinearity equations, two for every measured image points. Everything in these equations is unknown save the focal length and the coordinates of the control points, that serve to solve the system.

Example: Let a photograph have a focal length of 152.912 mm, and within it a point a has been observed with coordinates $(-15.713, -9.423)$. The approximate object coordinates of this point are

$$4939.978, 4251.036, 82.137$$

and the approximate values of the outer orientation parameters of the photograph are

$$\mathrm{O} \approx (5100, 4350, 1580)$$

$$\mathrm{M} \approx \begin{pmatrix} 0.98569 & 0.16859 & 0.00000 \\ -0.16859 & 0.98569 & 0.00000 \\ 0.00000 & 0.00000 & 1.00000 \end{pmatrix}$$

It is required to establish the residual equations for the x and y photo coordinates of this point.

According to the colinearity equations, the photo coordinates corresponding to the given approximate values and the known focal length are $(-17.806, -7.204)$; and the partial derivatives of these coordinates with respect to each of the parameters involved (with exception of the focal length, which we are taking fixed) are

$$\frac{\partial x}{\partial \mathrm{X_O}} = -0.10, \qquad \frac{\partial x}{\partial \mathrm{X_A}} = 0.10, \qquad \ldots \qquad \frac{\partial x}{\partial \omega} = 0.8, \ \ldots$$

$$\frac{\partial y}{\partial \mathrm{X_O}} = 0.02, \qquad \frac{\partial y}{\partial \mathrm{X_A}} = -0.02, \qquad \ldots \qquad \frac{\partial y}{\partial \omega} = 153, \ \ldots$$

The complete residual equations are

$$v_x = -0.10\Delta\mathrm{X_O} - 0.02\Delta\mathrm{Y_O} + 0.01\Delta\mathrm{Z_O} + 0.8\omega - 155\phi + 7.2\kappa \\ + 0.10\Delta\mathrm{X_A} + 0.02\Delta\mathrm{Y_A} - 0.01\Delta\mathrm{Z_A} - 2.092$$

$$v_y = 0.02\Delta\mathrm{X_O} - 0.10\Delta\mathrm{Y_O} + 0.0\Delta\mathrm{Z_O} + 153\omega - 0.8\phi - 18\kappa \\ - 0.02\Delta\mathrm{X_A} + 0.10\Delta\mathrm{Y_A} + 0.0\Delta\mathrm{Z_A} - 2.219$$

If the focal length where also being calibrated the above given value would just be an approximate value like the others, and a term corresponding to Δf would be present in both equations.

The coordinates of the control points may also be adjusted. In that case the difference between the adjusted value and the observed one is to be taken as a residual:

$$v_{\mathbf{X}} = \mathbf{X}_{\mathrm{aj}} - \mathbf{X}_{\mathrm{ob}}$$

Suppose the values ω, ϕ, κ are found to be

$$\omega = 0.0044, \quad \phi = -0.0126, \quad \kappa = -0.035 \quad (rad)$$

Find the new matrix M.

According to parametrization 6 of "Matrices de rotación" we successively find the values:

$$\omega^2 = 19\cdot 10^{-6}, \quad \phi^2 = 159\cdot 10^{-6}, \quad \kappa^2 = 0.00123, \qquad p = 0.999649;$$

$$\tfrac{1}{2}\omega\phi = -28\cdot 10^{-6}, \quad \tfrac{1}{2}\phi\kappa = 221\cdot 10^{-6}, \tfrac{1}{2}\kappa\omega = -77\cdot 10^{-6},$$

$$\mathrm{M}_\Delta = \begin{pmatrix} 0.99931 & 0.03496 & -0.01267 \\ -0.03502 & 0.99938 & -0.00418 \\ 0.01252 & 0.00462 & 0.99999 \end{pmatrix}$$

and finally

$$\mathrm{M}_\Delta \mathrm{M} \to \mathrm{M} = \begin{pmatrix} 0.97911 & 0.20293 & -0.01267 \\ -0.20300 & 0.97917 & -0.00418 \\ 0.01156 & 0.00666 & 0.99999 \end{pmatrix}$$

If parametrization 5 is used instead, the greatest difference with respect to the current solution is just $6\cdot 10^{-6}$ at places m_{12} and m_{21}.

There can also exist direct measurements of the coordinates and rotation matrices of points: GPS and INS observations. These observations are like control points, in the sense that they constitute direct measurements of parameters from the adjustment. However, they may bear constant errors, so that the equation of the observed GPS coordinates would be

$$\mathbf{X}_{\mathrm{GPS}} = \mathbf{X}_{\mathrm{O}} + \mathbf{X}_{\mathrm{shift}}$$

Furthermore, this constant error changes from one strip to another, so a different vector for each strip has to be computed.

The equation of the observed INS matrix is

$$\mathrm{M}_{\mathrm{INS}} = \mathrm{M}_{\mathrm{shift}}\mathrm{M}$$

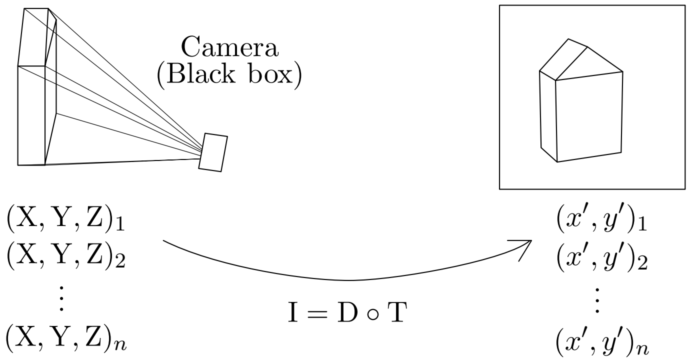
There can be higher order error terms (i.e., linear), but they usually reflect the fact that the GPS or INS coordinates have not been properly computed.

Photograph with distortion

Two expositions of this subject can be found in the documents "Elementos de calibración de una proyección central" and "A general purpose geometric distortion model for central projection cameras". The first one is a partial, easier to read exposition. It tacitly assumes that the distortion at the principal point is zero. This point is explained in detail in the second one, which is a precise comprehensive exposition of the geometric-mathematical aspects together with the proposal of a general set of parameters.

The starting point of the study of the photograph with distortion is as follows:

fig. 14



There is an object, which from the orientation viewpoint is a set of object coordinates: $(\mathrm{X},\mathrm{Y},\mathrm{Z})_1$, ...$(\mathrm{X},\mathrm{Y},\mathrm{Z})_n$; there is a camera containing a support (a piece of film or a pixel matrix) where the rays that enter the camera are condensed and impressed thereon somehow; finally, there is the image taken by the camera, which from the orientation viewpoint is the set of image coordinates corresponding to the known object coordinates: $(x', y')_1$, ..., $(x', y')_n$. Of these three elements, the intermediate one, the camera, we know nothing of it. We don't know where it was positioned at the time of taking the image, nor its focal length and other internals.

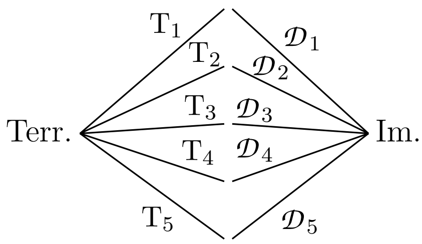
Hence, what we have is a set of $(\mathrm{X},\mathrm{Y},\mathrm{Z})$ object points together with their corresponding (x', y') images. By relating both sets we can guess where the camera was, $(\mathrm{X},\mathrm{Y},\mathrm{Z})_\mathrm{O}$, as well as the other six orientation parameters. But a camera is not a perfect machine, and therefore the (x', y') coordinates do not match any central projection whatsoever (this is why the prime index is used for them). Instead, the passage from the object coordinates to the image ones is the composition of two steps:

$$\begin{matrix}\text{Obj.}\\(\mathrm{X},\mathrm{Y},\mathrm{Z})\end{matrix} \xrightarrow{\mathrm{T}} (x, y) \xrightarrow{\mathrm{D}} \begin{matrix}\text{Im.}\\(x', y')\end{matrix}$$

A central projection *plus* a distortion. Of course, there isn't such a two step process inside the camera when the image is formed, but the meaning of this interpretation is that the transformations that carries each $(\mathrm{X},\mathrm{Y},\mathrm{Z})$ point to its corresponding (x', y') can be expressed as the composition of a perfect central projection, T, plus a distortion, $\mathcal{D}$.

But there happens that this decomposition is not unique —.i.e., if $\mathrm{I} = \mathrm{D}_1 \circ \mathrm{T}_1$ then also

$$\mathrm{I} = \mathrm{D}_1 \circ \mathrm{T}_1 = \mathrm{D}_2 \circ \mathrm{T}_2 = \mathrm{I} = \mathrm{D}_3 \circ \mathrm{T}_3 = \cdots$$



Some sections before it was shown that an image can be analytically changed into another one with different orientation parameters, provided the projection center is kept fixed. This degree of freedom is what causes an imperfect image to be decomposable in projection plus distortion in different ways. A detailed explanation is given in the second of the above mentioned works.

In case a camera is calibrated at laboratory by means of collimators, the object is not exactly a set of coordinates but a set of light rays concurrent at the projection center of the camera, forming known angles among themselves. Let there be for instance two arrays of collimators arranged in two perpendicular rows, intersecting at the central collimator, and, within each of the four lines starting at the central collimator, let the others be placed at angles of $7°30'$, $15°$, $22°30'$, $30°$, $37°30'$ and $45°$ with respect to the projection center. A total of 25 collimators.

It makes no sense to state that the central collimator is directed along the principal axis of the camera, and hence that its image is the principal point, for it would imply that the principal point is known, and it precisely that (among other parameters) which is being calibrated.

The paper “Elementos de calibración...” explains how to compute from this data the principal point (the point from where the distortion is most symmetric) and the focal length that best fits the image (the one that makes the distortions smaller). Here is an example.

Example: The distances of the collimator images to the central one along each of the four semi-diagonals are:

	$7°30'$	$15°$	$22°30'$	$30°$	$37°30'$	$45°$
s.d. 1	20.081	40.879	63.203	88.101	117.070	152.486
s.d. 2	20.082	40.876	63.202	88.096	117.069	152.483
s.d. 3	20.082	40.874	63.200	88.096	117.066	152.480
s.d. 4	20.079	40.875	63.199	88.095	117.063	152.479
average	20.081	40.876	63.201	88.097	117.067	152.482

The pairs of opposite semi-diagonals are 1/3 and 2/4.

We note that the computations of the principal point and of the focal length are independent of each other, for the later uses the average values along the four semi-diagonals and the former the difference between a semi-diagonal and its opposite one, and each of these quantities is not altered by a variation of the other parameter.

We shall begin with the principal point. The cited paper displays a formula to obtain the principal point, and within it the difference between distortions at equal angles. But this is the same as the difference of the real distances, so there is no need to compute approximate values and the corresponding distortions.

The differences s.d. 1 minus s.d. 3 and s.d. 2 minus s.d. 4 are

	$7°30'$	$15°$	$22°30'$	$30°$	$37°30'$	$45°$
$1-3$	-0.001	0.005	0.003	0.005	0.004	0.006
$2-4$	0.003	0.001	0.003	0.001	0.006	0.004

Let the difference corresponding to the angle α be called d_α. The formula for ϵ according to "Elementos de calibración..." is

$$\varepsilon = -\frac{\sum \tan^2\alpha\, d_\alpha}{2\sum \tan^4\alpha}$$

When applied to 1/3 this gives

$$\varepsilon_1 = -\frac{0.02\cdot(-0.001) + \cdots + 1\cdot 0.006}{2\cdot 3.0} = -0.004$$

And for the other diagonal,

$$\varepsilon_2 = -0.003$$

If the s.d. 1 follows the diagonal of the first quadrant, the s.d. 2 that of the second one and so forth, then

$$\varepsilon_x = \frac{\varepsilon_1 - \varepsilon_2}{\sqrt{2}} = -0.001$$
$$\varepsilon_y = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}} = -0.005$$

Suppose that the image of the central collimator has image coordinates $(x^c, y^c) = (130.020, 110.988)$; the calibrated principal point is therefore

$$x_p^c = 130.019, \quad y_p^c = 110.983$$

We proceed now to the computation of the focal length. It should be noted first that the angles that the collimators form with respect to the principal ray of the projection are not $7°30'$, etc., for the central collimator does not follow the direction of that ray. However, the difference with respect to these angles is very small and of opposite value for opposite semi-diagonals, i.e., if a collimator forms an angle of, say, $15°0'10''$ with the central ray, the opposite collimator will be at an angle of $14°59'50''$ from that same ray. The distance between the two images will very nearly be the distance between two opposite collimators at 15°. Hence, we can take half that distance as being the distance to the center corresponding to a collimator at 15°; but half that distance is equal to the average of the distances of both collimators to any intermediate point, such as the central collimator. Therefore, the mean distance to the central collimator of the four collimators forming a certain angle with it can be taken too as the mean distance to the principal point of the rays forming that same angle with the principal ray, and the calculus of the focal length is independent of the determination of the principal point, as claimed before.

The formula we will use now is not that of "Elementos de calibración...", but one which is simpler and does not require the previous computation of an approximate focal length and the corresponding distortions. This formula is

$$\mathrm{f} = \frac{\sum r'^2}{\sum r' \tan\alpha}$$

where by r' we mean the real distances (obviously, since the theoretical ones are not yet known).

The result for the current data is

$$\mathrm{f} = \frac{50784.0}{333.95} = 152.531$$

It is a remarkable fact that the principal point and the focal length can be computed without the need to have an approximate value for the later nor to even compute the distortions. But certainly the distortions are of interest. The small displacement of the central collimator with respect to the principal point of 0.004 and 0.003 along one and the other diagonals correspond to angles of $24\cdot 10^{-6}$ and $19\cdot 10^{-6}$ radians according to the computed focal length. These values equal $5''$ and $4''$ respectively. Therefore, the angles that the collimators form with the central ray of the projection are

s.d. 1	$7°30'05''$	$15°00'05''$	$22°30'05''$	$30°00'05''$	$37°30'05''$	$45°00'05''$
s.d. 2	$7°30'04''$	$15°00'04''$	$22°30'04''$	$30°00'04''$	$37°30'04''$	$45°59'04''$
s.d. 3	$7°29'55''$	$14°59'55''$	$22°29'55''$	$29°59'55''$	$37°29'55''$	$44°59'55''$
s.d. 4	$7°29'56''$	$14°59'56''$	$22°29'56''$	$29°59'56''$	$37°29'56''$	$44°00'56''$

and the distances of their images to the principal point:

s.d. 1	20.085	40.883	63.207	88.105	117.074	152.490
s.d. 2	20.085	40.879	63.205	88.099	117.072	152.486
s.d. 3	20.078	40.870	63.196	88.092	117.062	152.476
s.d. 4	20.076	40.872	63.196	88.092	117.060	152.476

The theoretical distances for each α are $\mathrm{f} \tan\alpha = 152.531 \tan\alpha$:

s.d. 1	20.085	40.874	63.185	88.069	117.047	152.538
s.d. 2	20.084	40.874	63.184	88.068	117.046	152.537
s.d. 3	20.077	40.867	63.176	88.059	117.035	152.524
s.d. 4	20.078	40.868	63.177	88.060	117.037	152.525

and finally the distortions and mean distortions for the different distances:

	20.1	40.9	63.2	88.1	117.0	152.5
s.d. 1	0.000	0.008	0.022	0.036	0.027	−0.049
s.d. 2	0.001	0.005	0.021	0.031	0.026	−0.051
s.d. 3	0.001	0.004	0.020	0.033	0.027	−0.047
s.d. 4	−0.002	0.005	0.019	0.032	0.024	−0.049
average	0.000	0.005	0.021	0.033	0.026	−0.049

There can be applied weight functions to the different distances. The reason is that some collimators represent a larger area of the photograph, and hence of the object, than others. The ones representing a smaller area are those nearest and furthest to the center. A weight function is provided in “A general purpose...”. A simple and correct solution is to weigh the innermost ones with 0.5 and the outermost with 0.7. The greater weight of the later with respect to the former is because the second outermost collimators have their images at a considerable distance from the outermost ones’, and therefore these represent a greater area than the innermost.

With these weights, the value for the principal point does not suffer a significant change but the focal length is now 152.539, and the average distortions

	20.1	40.9	63.2	88.1	117.0	152.5
s.d. 1	−0.001	0.003	0.017	0.029	0.020	−0.057

Photographing the stars

There is in theory no more exact method of calibrating a camera than taking a picture to the starry sky, for there is not any set of points anywhere of which a photograph can be taken with a position better known than the celestial objects.

However, it is not exempt from problems. The first one is obvious, the photograph has to be taken at night, with a clear sky and from a point with enough visibility. But there are more. In order that the stars appear on the photograph, a long exposure time is necessary. But it cannot be too long, for then the movement of the start would be significant: one second of time equals $360/24 = 15$ arc seconds, so four seconds is already $1'$. The problem is not so much the movement of the image during the exposure, since it is a regular displacement and the midpoint of the images will always correspond to the mean time of the exposition; as it is the fact that if the image keeps moving form pixel to pixel then no pixel will receive enough light to be activated.

There is also the problem that if we want a fine calibration, and hence need hundreds of stars to appear, a longer or more sensible exposure is required.

Of all the difficulties mentioned so far, the only one that actually matters is that of being able at all to get a picture with stars enough for the calibration, due to the problem of the faint light of them. The others are ridiculous compared to the effort of building and measuring a high precision calibration set of points, and the advantages with respect to the traditional method are enormous: no need to maintain (i.e., keep intact and occupying space, and measure from time to time) the calibration grid; no need to be at the laboratory; more precision than we will ever need; automatic calibration. For outdoors terrestial applications the benefits are apparent —take a picture the night before the starting of the job and another one at the end.

With respect to automatic calibration, this is so because the position of stars is a well known ever fixed data, that shall be stored by the calibrating program itself. Furthermore, the software can compare the picture with its known sky and automatically identify the stars, whence performing the calibration. Only an approximated date is needed for the dozen or so stars with relevant per year movement and for the correction to apply because of Earth velocity in its movement around the Sun and the subsequent apparent displacement of the celestial objects, all these done by the software.

So here are the formulae. There is no need to derive equations anew, for in the equations of colinearity, numerator and denominator can be divided by any and the same constant. They may in particular be divided by $\sqrt{(\mathrm{X}-\mathrm{X}_0)^2+(\mathrm{Y}-\mathrm{Y}_0)^2+(\mathrm{Z}-\mathrm{Z}_0)^2}$, in which case the equations become

$$
\begin{aligned}
x &= -\mathrm{f}\,\frac{m_{11}\alpha+m_{12}\beta+m_{13}\gamma}{m_{31}\alpha+m_{32}\beta+m_{33}\gamma}\\
y &= -\mathrm{f}\,\frac{m_{11}\alpha+m_{12}\beta+m_{13}\gamma}{m_{31}\alpha+m_{32}\beta+m_{33}\gamma}
\end{aligned}
\tag{3}
$$

α, β, γ being the cosinus of the light ray with the X, Y and Z axes respectively.

These equations capture the essence of the projection process and geometry better than the usual version does, for they reflect the fact that the image does not exactly depend on the position of the object point, but rather of the direction where it lies with respect to the projection center, given by the α, β, γ quantities, which are not three independent parameters but just two. And they reflect what was said before that different images can be computed out of a given one with different projection parameters as long as the projection center is fixed, because if this point is respected the projective rays, *i.e.,* α, β, γ, do not change.